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$N = 1, D = 6$ Supergravity: Duality and non Minimal Couplings.

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Abstract

Six-dimensional supergravity theories and their duality properties play a central role in the context of string duality and string compactifications. Lowering dimensions leads usually to an increasing complexity of theories; with this respect six dimensions seem to constitute an appropriate compromise between the physical four and the presumably more fundamental ten or eleven dimensions. In this paper we present a superspace formulation of $N = 1, D = 6$ supergravity with one tensor-multiplet and an arbitrary number of vector- and hypermultiplets, in which the bosonic abelian superforms of the theory, the dilaton, the abelian gauge fields and the two-form are replaced by their S-duals i.e. four, three and two-superforms respectively, in compatibility with supersymmetry. As usual this replacement interchanges Bianchi identities with equations of motion. This formulation holds in the presence of one tensor multiplet and arbitrary numbers of hypermultiplets and abelian super-Maxwell multiplets if all couplings are minimal. We determine the consistency conditions for non-minimal couplings in $N = 1$, $D = 6$ supergravity, for which we present a particularly significant solution, namely the one associated with the Chern-Simons-Lorentz three-form which entails the Green-Schwarz anomaly cancellation mechanism. In the case of non minimal couplings it is found that the gauge fields and the two-form can still be dualized while the dilaton has to remain a zero-form.

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1 Introduction

Borne as an attempt of quantizing gravity, supergravity theories in diverse dimensions play nowadays an important role as low-energy effective field theories of superstring and membrane theories. Supergravity theories have been extensively studied in four dimensions, of course because of their direct physical relevance, and in ten and eleven dimensions because of their fundamental features.

Even if it is known that consistent superstring theories can be formulated in six dimensions and that six-dimensional supergravity can arise as their low-energy limit, this is not the only reason for investigating $D = 6$ supergravity. In fact, while $D = 10$ and $D = 11$ theories are of direct interest as backgrounds for strings, membranes and M-theory, one frequently performs compactifications down to $D = 6$ to clarify relations among these theories, which are hidden in their ten or eleven-dimensional formulations.

Actually, the greatest recent step forward in understanding superstring theories was the discovering of *duality*. All known superstring theories, membrane theories and M-theory are related by duality and can be regarded as expansions of a unique theory around different vacua. Most of the times these relations are evident only after compactification, frequently to six dimensions.

For example, it has been conjectured [1] that compactification of the type IIA theory on K_3 and the heterotic string on $\mathbb{T}^4$ gives rise to the same theory on the remaining six-dimensional space and in [2] it has been shown heuristically that this equivalence can be understood as a consequence of M-theory. Another example is M-theory compactified on $(K_3 \times S_1)/\mathbb{Z}_2$ (see [3]) which gives rise to new couplings in $N = 1, D = 6$ supergravity which have been studied in [4].

Moreover, recently a possible new fundamental theory (F-theory) living in a twelve–dimensional space with signature $(10, 2)$ has been conjectured in [5]. As shown in [5], compactifications of F-theory on Calabi-Yau manifolds should give rise to supergravity theories in $D = 6$ with $N = 1$ supersymmetry. Some details of these compactifications are available in [6].

For more details on the relevance of six–dimensional supergravity theories see for example the introduction in [7].

In this paper we are concerned with the Hodge S-duality of p–forms in six-dimensional supergravity addressing the problem of the dualizability of these forms in compatibility with supersymmetry. While for $D = 11$ supergravity this analysis has been performed in [8, 9], for ten dimensions in [10] the authors presented a formulation of $N = 1, D = 10$ supergravity super-Maxwell theory in superspace in which *all the bosonic abelian fields* can be described as p-superforms or as $(D-p-2)$-superforms, the bosonic components of the supercurvatures of these fields being related by Hodge-duality. For the corresponding results in IIA and IIB supergravity see ref. [11]. The conjecture that emerges from these papers is that in every Supergravity theory and in every dimension an abelian form can

be described alternatively in either of the two ways in compatibility with supersymmetry.

The results of the present paper extend the validity of this conjecture also to the six–dimensional case. Actually, these features hold true if the couplings are all minimal. Interesting non minimal couplings arise (in two, six and ten dimensions) from a Lorentz–Chern–Simons term in the two-form curvature, which realizes the Green–Schwarz anomaly cancellation mechanism. A supersymmetric version of this mechanism [12, 13, 14] amounts to a supersymmetrization of the Lorentz-Chern-Simons form or equivalently to the solution of the superspace Bianchi identity

$$dH = c_1 trF^2 + c_2 trR^2 \tag{1.1}$$

for $c_2 \neq 0$. This problem has been solved in ten dimensions through the so called Bonora-Pasti-Tonin theorem [12]. In this paper we prove the six–dimensional version of this theorem which guarantees essentially the solvability of (1.1) in superspace. This allows us to test the duality conjecture also in the presence of the non-minimal couplings arising from the trR^2 term in (1.1). The principal result we found is that all the abelian forms but the dilaton can still be dualized.

In this paper we deal with $N = 1, D = 6$ supergravity with one tensor multiplet, an arbitrary number of vector multiplets and hypermultiplets in a superspace formalism [15, 16, 17, 18]. In section two we present our superspace conventions and notations.

Section three is devoted to the solution of the relevant superspace Bianchi identities (B.I.). First of all we perform a group theoretical analysis of the superspace constraints. This allows us first to classify them and, second, to derive consistency conditions on the remaining undetermined auxiliary fields which trigger the couplings between the multiplets: each choice for the auxiliary fields satisfying these consistency conditions gives rise to a different theory.

A more precise analysis of these consistency conditions shows that even with minimal couplings we are not dealing with a unique theory but with a *family of theories*. For these theories the couplings depend on a real parameter k which cannot be scaled away. As one sends $k \to 0$ one obtains the standard minimal couplings between all the multiplets, while for $k \to \infty$ one obtains the coupled tensor + Yang-Mills system in flat space, i.e. the supergravity multiplet decouples. This should present the limiting procedure missed in [7].

Having solved the fundamental B.I.'s, in section four we construct superforms dual respectively to the dilaton, Maxwell gauge fields, and the two-form B, such that they satisfy dual B.I.'s in superspace. These represent the equations of motion for the basic abelian forms and allow, in turn, to define dual super-potentials, i.e. four, three and two-superforms respectively. The basic B.I.'s can then be interpreted as equations of motion for the dual potentials.

In this way we give a strong support to the conjecture that every abelian p-superform in a supergravity theory can indeed be described by a dual $(D-p-2)$-superform although

a general proof of this conjecture is still missing.

In section four the analysis was performed for simplicity in the absence of hypermultiplets, so in section five we extend the above mentioned results to the case where the hypermultiplets are present. It remains an open question whether the scalars belonging to the hypermatter can themselves be transformed to four-forms (see however [18]).

Section six is devoted to the supersymmetrization of the Lorentz–Chern–Simons form, which presents a non–minimal solution of the above mentioned consistency conditions, and to an analysis of the duality properties of the resulting theory.

Section seven contains some concluding remarks while in the appendix we give more details on our notations and a few fundamental gamma matrix identities.

2 Preliminaries

Six-dimensional $N = 1$ supergravity allows for four kind of multiplets:

$$\text{Pure supergravity} \qquad \{e_m^a, \psi_m^{\alpha i}, B_{ab}^-\}, \tag{2.1}$$

$$\text{Tensor multiplet} \qquad \{B_{ab}^+, \lambda_{\alpha i}, \phi\}, \tag{2.2}$$

$$\text{Yang-Mills} \qquad \{A_a, \chi^{\alpha i}\}, \tag{2.3}$$

$$\text{Hypermultiplet} \qquad \{\psi_\alpha^Y, \varphi^I\}. \tag{2.4}$$

The superspace in six dimensions is spanned by the supercoordinates $Z^M = (x^m, \vartheta^{\mu i})$ where x^m $(m = 0, \ldots, 5)$ are the ordinary space-time coordinates and $\vartheta^{\mu i}$ $(\mu = 1, \ldots 4)$ are symplectic Majorana-Weyl spinors carrying the $USp(1)$ doublet index $i = 1, 2$. In what follows letters from the middle of the alphabet represent curved indices while letters from the beginning represent flat indices: small Latin letters $a = (0, \ldots, 5)$ indicate vectorial indices, small Greek letters $\alpha = (1, \ldots, 4)$ indicate spinorial indices and capital letters denote both of them $A = (a, \alpha i)$. The fields ϕ^I $(I = 1, \ldots, 4n_H)$ constitute the coordinates of the quaternionic Kähler manifold $USp(n_H, 1)/USp(n_H) \otimes Usp(1)$, where n_H is the number of hypermultiplets, and the index $Y = 1, \ldots, 2n_H$ stands for the fundamental representation of $USp(n_H)$.

The superspace geometry is described by the vielbeins $E^A = dZ^M \; {E_M}^A(Z)$, the Lorentz–valued connection ${\Omega_A}^B$, the Lie-algebra-valued Yang-Mills (YM) connection A, which are one-superforms, and the two-superform B. We remember that a superfield ${\psi_A}^B$ is Lorentz-valued if ${\psi_a}^b = -{\psi^b}_a$, ${\psi_\alpha}^\beta = \frac{1}{4}\, {(\Gamma^{ab})_\alpha}^\beta \; \psi_{ab}$ and vanishes otherwise. Since our spinors are four component Weyl spinors we use a Weyl algebra of 4×4 Γ matrices

$$(\Gamma^a)_{\alpha\beta}(\Gamma^b)^{\beta\gamma} + (\Gamma^b)_{\alpha\beta}(\Gamma^a)^{\beta\gamma} = 2\, \eta^{ab}\, \delta_\alpha^\gamma. \tag{2.5}$$

$\Gamma^{a_1 \ldots a_n}$ will denote completely antisymmetrized product of Γ matrices, where antisymmetrization is understood with unit weight.

A p-superform can be decomposed in the vielbein basis as

$$\phi_p = \frac{1}{p!} E^{A_1} - E^{A_p} \phi_{A_p - A_1}(Z). \tag{2.6}$$

It will be useful to call $(q, p-q)$ sector of ϕ_p, denoted by $\phi_{(q,p-q)}$, the component of ϕ_p proportional to q vector-like supervielbeins E^a and $p-q$ spinor-like supervielbeins $E^{\alpha i}$.

The torsion T^A and the Lorentz and Yang-Mills curvatures $R_A{}^B$ and F are defined as

$$T^A = DE^A = dE^A + E^B \Omega_B{}^A = \frac{1}{2} E^B E^C T_{CB}{}^A, \tag{2.7}$$

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B = \frac{1}{2} E^C E^D R_{DCA}{}^B, \tag{2.8}$$

$$F = dA + AA = \frac{1}{2} E^B E^C F_{CB}, \tag{2.9}$$

while the B-field strength H depends on the model. In general one sets

$$H = dB + c_1 \omega_{3YM} + c_2 \omega_{3L}, \tag{2.10}$$

where ω_{3YM} is the YM Chern-Simons 3-super-form which couples Yang-Mills to the supergravity and tensor multiplet, while ω_{3L} is the Lorentz Chern-Simons form:

$$d\omega_{3YM} = trF^2, \qquad d\omega_{3L} = trR^2. \tag{2.11}$$

We define the Hodge duality relation between tensors as

$$\tilde{W}_{a_1 \ldots a_k} \equiv \frac{(-1)^{\frac{k}{2}(k+1)}}{(6-k)!} \epsilon_{a_1 \ldots a_6} W^{a_{k+1} \ldots a_6} \tag{2.12}$$

which allows to decompose antisymmetric three-tensors in self-dual (+) and anti-selfdual (−) parts

$$W^{(\pm)}_{abc} \equiv \frac{1}{2}(W_{abc} \pm \tilde{W}_{abc}), \tag{2.13}$$

which fulfill the Γ-projections:

$$(\Gamma^{abc})_{\alpha\beta} \, W^-_{abc} = 0, \qquad (\Gamma_{abc})^{\alpha\beta} \, W^{+abc} = 0. \tag{2.14}$$

Other notations we use in what follows are $[\ldots)$, $[\ldots]$,$(\ldots)$ on indices denoting graded symmetrization, antisymmetrization and symmetrization respectively, with unit weight.

3 The Bianchi Identities

The super–torsion and super–curvatures satisfy the Bianchi identities

$$\begin{aligned} DT^A &= E^B R_B{}^A, && (3.1) \\ DR_A{}^B &= 0, && (3.2) \\ DF &= 0, && (3.3) \\ dH &= c_1 tr(F^2) + c_2 tr(R^2). && (3.4) \end{aligned}$$

In this section we set $c_2 = 0$ and consider $c_2 \neq 0$ in section six where we discuss the supersymmetrization of the Lorentz–Chern–Simons form.

The unphysical fields present in the supercurvatures are eliminated by imposing suitable constraints. Once these constraints are imposed the B.I.'s are no longer identities and have to be solved consistently. One consistency requirement is the closure of the supersymmetry (SUSY) algebra:

$$D_A D_B - (-1)^{AB} D_B D_A = -T_{AB}{}^C D_C - R_{AB\#}{}^{\#} - [F_{AB},). \tag{3.5}$$

3.1 Constraint analysis

In the superspace language the dynamics of the physical fields is governed by the constraints one imposes on the supercurvatures. To classify the possible consistent sets of constraints for the torsion we adopt a general strategy which is based on group theoretical arguments, see e.g. [8, 14].

Our starting point is the fundamental rigid supersymmetry preserving constraint

$$T_{\alpha i \beta j}{}^a = -2\epsilon_{ij} \Gamma^a_{\alpha\beta}. \tag{3.6}$$

Once this constraint has been fixed the remaining ones fall into two classes: constraints which can be obtained by superfield redefinitions (kinematical constraints) and constraints which determine the dynamics and the couplings between the fields (dynamical constraints). The basic redefinitions are [14]

$$\begin{aligned} E'^{\alpha i} &= E^\alpha + E^b H_b{}^{\alpha i}, \\ \Omega'_{\alpha i a}{}^b &= \Omega_{\alpha i a}{}^b + X_{\alpha i a}{}^b, \end{aligned} \tag{3.7}$$

where $H_b{}^{\alpha i}$ and $X_{\alpha i a}{}^b$ are suitable covariant superfields with $X_{\alpha i ab} = -X_{\alpha i ba}$.

With these redefinitions we can eliminate from $T_{\alpha i a}{}^b$ all irreducible representations (irrep) of $SO(1,5)$ apart from the $\overline{60}$ which is symmetric in its vectorial indices ab and has all its Γ-traces vanishing. At this point the lowest sector of the torsion B.I.'s

$$T_{\alpha i \beta j}{}^{\sigma l} T_{\sigma l \gamma k}{}^a + T_{\alpha i \beta j}{}^b T_{b \gamma k}{}^a + (\text{cyclic permutations}) = 0, \tag{3.8}$$

implies that also $\overline{60}$ vanishes and that $T_{\alpha i\beta j}{}^{\gamma k}$ can at most contain three spinors, so

$$T_{\alpha i a}{}^{b} = 0, \tag{3.9}$$

$$T_{\alpha i\beta j}{}^{\gamma k} = \delta_\alpha{}^\gamma \omega_{\beta i,j}{}^{k} + (\alpha i \leftrightarrow \beta j), \tag{3.10}$$

where $\omega_{\beta i,jk}$, symmetric in j and k, describes the three spinorial degrees of freedom left.

The choice for ω corresponds to a dynamical constraint [17]: setting it to zero amounts to a decoupling of the hypermultiplets, while identifying ω with a $USp(1)$ connection and promoting the global $USp(1)$ invariance of the SUSY algebra in $D = 6$ to a local one leads to a natural description [15] of the self-interactions between the hypermultiplets on a quaternionic manifold. Since we discuss hypermultiplets in section five, here we set $\omega_{\alpha i,jk} = 0$.

The next step is to solve the B.I.'s with canonical dimension $d = 1$. These B.I.'s admit a unique solution modulo the connection shift

$$\Omega'_{ab}{}^{c} = \Omega_{ab}{}^{c} + Y_{a,b}{}^{c} \tag{3.11}$$

where $Y_{a,bc} = Y_{a[bc]}$ which allows to set the torsion $T_{ab}{}^{c}$ equal to an arbitrary tensor.

For a suitable $Y_{a,b}{}^{c}$ one obtains as general solution for the dimension 1 B.I.'s:

$$T_{abc} = T^{-}_{abc}, \tag{3.12}$$

$$T_{a\alpha i}{}_{j}^{\beta} = 3\delta_\alpha{}^\beta V_{aij} + \Gamma_{ac\alpha}{}^{\beta} V^{c}_{ij} - \epsilon_{ij}\frac{1}{4}\Gamma^{bc}{}_{\alpha}{}^{\beta} T^{-}_{abc}, \tag{3.13}$$

$$R_{\alpha i\beta jab} = -4\Gamma_{abc\,\alpha\beta} V^{c}_{ij}, \tag{3.14}$$

where T^{-}_{abc} is a completely antisymmetric anti-selfdual tensor and the vector $V_{a\,ij}$ is symmetric in i and j and constitutes an auxiliary field, which will play a central role in what follows.

The peculiarity of the choice (3.12)-(3.14) lies in the fact that for *pure* supergravity one has $V_{a\,ij} = 0$ (dynamical constraint), i.e.

$$R_{\alpha i\beta ja}{}^{b} = 0, \tag{3.15}$$

in analogy with the standard Yang-Mills constraint $F_{\alpha i\beta j} = 0$. The relation (3.15) allows, moreover, to apply a general cohomological technique developed in [19] for the determination of the (curved) supersymmetry anomalies which are inevitably associated to the ABBJ-$SO(1,5)$ local Lorentz anomalies in supersymmetric six-dimensional theories with chiral fermions (or bosons).

The $d = 3/2$ B.I.'s imply (and are solved by) the following three relations:

$$R_{a\alpha ibc} = -2\Gamma_{a\,\alpha\beta} T^{\beta}_{bci} + \Gamma_{[ab\,\alpha}{}^{\beta}\left(\frac{2}{3}\epsilon^{hk}\Gamma_{c]s\beta}{}^{\rho} D_{\rho h} V^{s}_{ki} - \frac{4}{3}\epsilon^{hk} D_{\beta h} V_{c]ki}\right), \tag{3.16}$$

$$\Gamma^{s}_{\alpha\beta} T_{sa}{}_{i}^{\beta} = \epsilon^{hk} D_{\alpha h} V_{a\,ki} - \frac{1}{3}\epsilon^{hk}\Gamma_{as\,\alpha}{}^{\beta} D_{\beta h} V^{s}_{ki} \tag{3.17}$$

$$D_{\alpha(i} V^{a}_{jk)} = 0. \tag{3.18}$$

The first relation parametrizes $R_{a\alpha i,bc}$ in terms of the gravitino field-strength $T_{ab}{}^{\alpha i}$ and of the spinorial derivatives of $V_{a\,ij}$ and the second one is the equation of motion for the gravitino. The third relation constitutes a consistency condition on the auxiliary field $V_{a\,ij}$: every choice one makes for $V_{a\,ij}$, implementing thus a dynamical constraint, has to satisfy (3.18) identically. A part from the trivial choice $V_{a\,ij} = 0$, which leads to pure supergravity (SUGRA), we will present solutions for (3.18) in the present and the next sections.

The solution for the F-B.I. proceeds as follows. The constraints on the lowest components of the Yang-Mills supercurvature F are the usual constraints proposed by Nilsson for the free theory [20]

$$F_{\alpha i\beta j} = 0. \tag{3.19}$$

Solving the F-B.I. with these constraints one finds that

$$F_{a\alpha i} = -2\,\Gamma_{a\,\alpha\beta}\,\chi_i^\beta, \tag{3.20}$$

where $\chi^{\alpha i}$ is the gluino superfield, together with the supersymmetry transformation rules for the gluon field-strength F_{ab} and the gluino

$$\begin{aligned} D_{\alpha i}F_{ab} &= 4\Gamma_{[b\ \alpha\rho}D_{a]}\chi_i^\rho + 4(\Gamma_{ba}\Gamma_c)_{\alpha\rho}V_{ij}^c\chi^{\rho j}, & (3.21)\\ D_{\alpha i}\chi_j^\beta &= \delta_\alpha{}^\beta A_{ij} - \epsilon_{ij}\frac{1}{4}\Gamma^{ab}{}_\alpha{}^\beta F_{ab}. & (3.22)\end{aligned}$$

Here A_{ij}, symmetric in i, j, is a Lie-algebra valued auxiliary superfield which remains undetermined by the B.I. themselves. However, under the closure of the SUSY algebra (3.5) on $\chi^{\alpha i}$, i.e. applying one spinorial derivative to (3.22) and using on its r.h.s. (3.21), one gets the consistency condition

$$D_{\alpha\,(i}A_{jk)} = -4\Gamma^a_{\alpha\beta}\,\chi^\beta_{(i}V_{a\,jk)}. \tag{3.23}$$

This is the constraint which will determine A_{ij} almost uniquely once a $V_{a\,ij}$ satisfying (3.18) has been found.

The constraints for the H-B.I.'s

$$dH = c_1 trF^2, \tag{3.24}$$

are also standard

$$\begin{aligned} H_{\alpha i\beta j\gamma k} &= 0, & (3.25)\\ H_{a\alpha i\beta j} &= -2\epsilon_{ij}\Gamma_{a\,\alpha\beta}\phi, & (3.26)\\ H_{ab\beta i} &= -\Gamma_{ab\,\beta}{}^\rho\lambda_{\rho i} & (3.27)\end{aligned}$$

where ϕ is the dilaton superfield and $\lambda_{\alpha i} \equiv D_{\alpha i}\phi$ is the gravitello. (3.24) implies also the supersymmetry transformations

$$\begin{aligned} D_{\alpha i}\lambda_{\beta j} &= 4\phi\Gamma_{c\alpha\beta}V_{ij}^c + 2c_1\Gamma_{c\alpha\beta}\chi_i\Gamma^c\chi_j + \epsilon_{ij}(\Gamma^a_{\alpha\beta}D_a\phi - \frac{1}{12}H^+_{\alpha\beta}). & (3.28)\\ D_{\alpha i}H_{abc} &= -3\Gamma_{[ab\,\alpha}{}^\rho D_{c]}\lambda_{\rho i} - 3\Gamma_{[ab\,\sigma}{}^\rho T_{c]\alpha}{}^\sigma\lambda_{\rho i} - 3\Gamma^s{}_{[a\,\alpha}{}^\rho T^-_{bc]s}\lambda_{\rho i} + \\ &+ 6\phi\Gamma_{[a\,\alpha\sigma}T_{bc]}{}^\sigma_i + 8c_1\Gamma_{ca\,\alpha\beta}tr(\chi_i^b F_{bc]}), & (3.29)\end{aligned}$$

where $H^+_{\alpha\beta} \equiv (\Gamma^{abc})_{\alpha\beta} H_{abc}$, and the relation

$$T^-_{abc} = \frac{1}{\phi} H^-_{abc} + \frac{c_1}{\phi} tr(\chi_i \Gamma_{abc} \chi^i), \tag{3.30}$$

which leads to the identification of T^-_{abc} as the B^-_{ab} curvature.

In all the cases we considered in this paper the closure of the SUSY algebra on $\lambda_{\alpha i}$ is automatic once (3.18) and (3.23) are satisfied.

It is useful to notice that the consistency condition (3.18) is equivalent to the existence of a tensor $X_{k\alpha}$ such that

$$D_{\alpha i} V^a_{jk} = \epsilon_{i(j} X_{k)\alpha}. \tag{3.31}$$

3.2 Consistency conditions

¿From the solution of the B.I.'s we found two auxiliary fields which permit to couple the various multiplets. Thus, $V_{a\,ij}$ and A_{ij} represent the only freedom left (a part from $\omega_{\alpha\,i,jk}$). We saw that these fields are restricted by the consistency conditions

$$\begin{aligned} D_{\alpha(i} V^a_{jk)} &= 0, & (3.32)\\ D_{\alpha(i} A_{jk)} &= -4\Gamma^a_{\alpha\beta} \chi^\beta_{(i} V_{a\ jk)}, & (3.33) \end{aligned}$$

for which we will give now the solution which leads to the minimally coupled SUGRA-TENSOR-YM theory. We make the Ansatz

$$\begin{aligned} V^a_{ij} &= \gamma(\phi) \lambda_i \Gamma^a \lambda_j + \delta(\phi) \chi_i \Gamma^a \chi_j, & (3.34)\\ A_{ij} &= \beta(\phi) \lambda_{(i} \chi_{j)}, & (3.35) \end{aligned}$$

where γ, δ and β are functions of ϕ which have to be determined from (3.32)-(3.33).

Inserting (3.34) and (3.35) in (3.32) and (3.33) we find that the consistency conditions are identically satisfied if and only if γ, δ and β satisfy the following set of coupled differential equations (where $\gamma' = \frac{d\gamma}{d\phi}$ etc.):

$$\begin{cases} \gamma' - 32\phi\gamma^2 = 0, \\ \delta' + \delta\beta = 0, \\ c_1\beta + 2\delta + 2\phi\delta\beta = 0, \\ \beta' - \beta^2 - 16\gamma(\phi\beta + 1) = 0. \end{cases} \tag{3.36}$$

The general solution of this set of equation is parametrized by a real parameter k and is given by

$$\begin{aligned} \gamma(\phi) &= -\frac{1}{k + 16\phi^2} & (3.37)\\ \beta(\phi) &= \frac{4}{\sqrt{16\phi^2 + k}} & (3.38)\\ \delta(\phi) &= -\frac{c_1}{2\phi + 2\sqrt{\phi^2 + \frac{k}{16}}}. & (3.39) \end{aligned}$$

The result (3.37)-(3.39) with the corresponding relations (3.34)-(3.35) for $V_{a\,ij}$ and A_{ij} generalizes the solutions for the B.I.'s (3.1)-(3.4) (with $c_2 = 0$) which one finds in the literature [17]. These solutions can be obtained from (3.37)–(3.39) in the following limits. For $k \to 0$ one gets

$$\gamma(\phi) = -\frac{1}{16\phi^2}, \quad \beta(\phi) = \frac{1}{\phi}, \quad \delta(\phi) = -\frac{c_1}{4\phi}, \tag{3.40}$$

while for $k \to \infty$ one obtains $\gamma = \beta = \delta = 0$. The corresponding expressions for the auxiliary fields are

$$k \to 0, \quad A_{ij} = \frac{1}{\phi}\lambda_{(i}\chi_{j)}, \quad V^a_{ij} = -\frac{1}{16\phi^2}\lambda_i\Gamma^a\lambda_j - \frac{c_1}{4\phi}\chi_i\Gamma^a\chi_j; \tag{3.41}$$

$$k \to \infty, \qquad A_{ij} = 0, \qquad V^a_{ij} = 0. \tag{3.42}$$

Equation (3.41) reproduces the standard minimally coupled SUGRA-TENSOR-YM system while (3.42) gives rise to the decoupled pure SUGRA and to the coupled (rigid) TENSOR-YM theory [7].

At present only the choice (3.41) gives rise to equations of motion which can be derived from an action [21].

The choice (3.42) leads to equations of motion which describe on one hand a free YM theory and on the other hand a tensor multiplet which is coupled, through the Chern-Simons term, to the YM multiplet. For this system the authors in [7] were able to derive an action which describes, however, not only the propagation of the TENSOR-YM system, but also the propagation of a spurious additional tensor multiplet.

A peculiarity of the tensor multiplet is the presence of a self-dual boson. Manifestly covariant actions for self-dual bosons have been constructed only recently [22]; it does, however, not seem that the form of these actions can be generalized to describe the TENSOR-YM system, the principal problem being that the YM equations of motion are free while the tensor equations of motion are not.

It is, however, possible to write a manifestly Lorentz-covariant and supersymmetric action for the free tensor multiplet and for the free supergravity multiplet [23].

For a generic k one obtains equations of motion, for a coupled SUGRA-TENSOR-YM system, which close among them under SUSY (just like for the TENSOR-YM system), but at present it is not clear to us if this system admits an action. This point deserves further investigation. Notice, however, that the limiting cases (3.41) and (3.42) are the unique cases which preserve scale-invariance.

Notice also that in this section we showed that the TENSOR-YM system can be obtained as a limiting case ($k \to 0$), from a coupled SUGRA-TENSOR-YM system, in which SUGRA decouples. This presents, in particular, the limiting procedure missed in [7].

Once one has an explicit solution for the consistency conditions, it remains to derive the equations of motion for the physical fields. Since for our purposes we do not need them

we will not give them here explicitly. While Einstein's equation and the equations for the fermions have to be derived always using standard superspace techniques the equations of motion for the abelian forms will be a byproduct of our duality technique, which we illustrate in the next section.

4 Duality

In supergravity theories, all the bosonic fields, apart from the metric, are described through forms. We described the dilaton ϕ with a 0-form, the gluon A with the gauge connection 1-form and the graviphoton B with a two-form, the curvatures associated to these potentials being $F = dA + AA$, $H = dB + c_1\omega_{3YM}$ and, implicitly, we used also the one–form curvature for the dilaton $V = d\phi$.

The number of bosonic physical degrees of freedom described by a p-form potential in D dimensions is given by

$$\begin{pmatrix} D-2 \\ p \end{pmatrix} = \frac{(D-2)!}{p!(D-p-2)!}. \tag{4.1}$$

which is manifestly invariant under $p \to (D-p-2)$. Therefore, as it is well known, at the kinematical level a p-form potential, with a $(p+1)$-form as curvature, carries the same degrees of freedom as a $(D-p-2)$-form potential with a $(D-p-1)$-form as curvature, the two curvatures being related by Hodge duality.

The problem we address here is if this duality is compatible (and to which extent) with supersymmetry. In the superspace language the solution of this problem (see [10]) amounts to determine supercurvatures $\tilde{F}_4$, $\tilde{H}_3$ and $\tilde{V}_5$, associated respectively to F_2, H_3 and V_1, in such a way that their bosonic components are Hodge-dual to the bosonic components of F_2, H_3 and V_1 respectively, and that $\tilde{F}_4$, $\tilde{H}_3$ and $\tilde{V}_5$ satisfy B.I.'s whose bosonic components are the equations of motion for the bosonic potentials A_1, B_2 and ϕ_0 we started with. Moreover, and most importantly, these new B.I.'s have to be "true" B.I.'s in superspace in the sense that they must allow to introduce new superpotentials $\tilde{A}_3$, $\tilde{B}_2$ and $\tilde{\phi}_4$ which, in turn, can be interpreted as superspace duals of A_1, B_2 and ϕ_0. The original B.I.'s for A_1, B_2 and ϕ_0 can then be read as equations of motion for $\tilde{A}_3$, $\tilde{B}_2$ and $\tilde{\phi}_4$.

In this section we will implement this program for the minimal SUGRA–TENSOR–SUPER MAXWELL–system (abelian gauge fields) and we will present its generalizations in the next section. The restriction to abelian gauge fields is necessary because in that case the gluons are decoupled from the gluinos, while in the non-abelian case the gluino current, appearing at the r.h.s. of the YM–equation of motion is "non-topological", in the sense that off–shell it can not be expressed as the differential of any local form. This prevents one from introducing dual non–abelian potentials $\tilde{A}_3$, see below. Therefore we restrict ourself to the gauge group $G = [U(1)]^N$ and $F \equiv (F_1, \ldots, F_N)$, where $F_i = dA_i$.

We recall that the standard B.I.'s are

$$
\begin{aligned}
dF &= 0 && (4.2)\\
dH &= c_1 trF^2 && (4.3)\\
dV &= 0. && (4.4)
\end{aligned}
$$

Our Ansatz for the B.I.'s of the dual superforms $\tilde{F}, \tilde{H}$ and $\tilde{V}$ is the following (see also [10] for the ten dimensional case)

$$
\begin{aligned}
d\tilde{F} &= F\tilde{H} && (4.5)\\
d\tilde{H} &= 0 && (4.6)\\
d\tilde{V} &= c_1 tr(F\tilde{F}) + H\tilde{H}. && (4.7)
\end{aligned}
$$

Notice first of all that this Ansatz is self-consistent in that, thanks to (4.2)-(4.4) the right hand sides of (4.5)-(4.7) are closed forms. Moreover, it is easy to see that the r.h.s. are also exact: it is this non trivial fact which will allow us, according to our above requirement, to introduce dual super potentials in section 4.1.

We will now give a constructive proof of (4.5)-(4.7) specifying the components of $\tilde{H}$, $\tilde{F}$ and $\tilde{V}$ which satisfy them identically. We define:

$$
\begin{aligned}
\tilde{H}_{\alpha i\beta j\gamma k} &= 0, && (4.8)\\
\tilde{H}_{a\alpha i\beta j} &= \frac{2}{\phi}\epsilon_{ij}(\Gamma_a)_{\alpha\beta}, && (4.9)\\
\tilde{H}_{ab\alpha i} &= -\frac{1}{\phi^2}(\Gamma_{ab})_\alpha{}^\beta\lambda_{\beta i}, && (4.10)\\
\tilde{H}_{abc} &= \frac{1}{\phi^2}\frac{1}{3!}\epsilon_{abcdef}\left(H^{def} - \frac{1}{4\phi}\lambda_i\Gamma^{def}\lambda^i + c_1 tr(\chi^i\Gamma^{def}\chi_i)\right), && (4.11)
\end{aligned}
$$

for $\tilde{H}$,

$$
\begin{aligned}
\tilde{F}_{(p,4-p)} &= 0 \qquad p \leq 2, && (4.12)\\
\tilde{F}_{abc\alpha i} &= -\frac{2}{\phi}(\Gamma_{abc})_{\alpha\beta}\chi_i^\beta, && (4.13)\\
\tilde{F}_{sabc} &= \frac{1}{2!\phi}\epsilon_{sabcef}\left(F^{ef} + \frac{1}{\phi}\chi^k\Gamma^{ef}\lambda_k\right), && (4.14)
\end{aligned}
$$

for $\tilde{F}$ and

$$
\begin{aligned}
\tilde{V}_{(p,5-p)} &= 0 \qquad p \leq 3, && (4.15)\\
\tilde{V}_{abcd\alpha i} &= \frac{2}{\phi}\Gamma_{a_1\ldots a_4\,\alpha}{}^\rho\lambda_{\rho i}, && (4.16)\\
\tilde{V}_{a_1\ldots a_5} &= -\frac{1}{\phi}\epsilon_{a_1\ldots a_6}\left(D^{a_6}\phi - \frac{1}{4\phi}\lambda_k\Gamma^{a_6}\lambda^k + 2c_1 tr(\chi_k\Gamma^{a_6}\chi^k)\right), && (4.17)
\end{aligned}
$$

for $\tilde{V}$. The proof that, with these definitions, the B.I's (4.5)-(4.7) are indeed satisfied goes as follows. First one proves (4.6). Defining the four–superform $K_4 \equiv d\tilde{H}$ it is easy to show that $K_{(0,4)} = \ldots = K_{(3,2)} = 0$. At this point one uses the following

Lemma. If a p-superform K_p, with $3 \leq p \leq 6$, satisfies

1. $dK_p = 0$,
2. $K_{(0,p)} = \ldots = K_{(p-2,2)} = 0$,

then $K_p = 0$ identically, (see [10, 19]).

Since $d\tilde{K}_4 = 0$ the lemma implies now that $\tilde{K}_4 = 0$ and (4.6) holds. If we define now the five-superform $K_5 \equiv d\tilde{F} - F\tilde{H}$, thanks to (4.6) and (4.2) we have $dK_5 = 0$ and it is straightforward to show that $K_{(0,5)} = \ldots = K_{(3,2)} = 0$. The lemma implies then that $K_5 = 0$ and also (4.5) holds. Finally (4.2), (4.3), (4.6) and (4.5) ensure now that $K_6 \equiv d\tilde{V} - c_1 tr(F\tilde{F}) - H\tilde{H}$ satisfies $dK_6 = 0$ and a direct calculation gives $K_{(0,6)} = \ldots = K_{(4,2)} = 0$. The lemma implies then also (4.7).

What we proved is essentially that (4.2)-(4.4) imply (4.5)-(4.7). But, since the purely bosonic components of the dual supercurvatures are defined essentially as the Hodge-duals of the basic bosonic curvatures (see equations (4.11), (4.14) and (4.17)), the purely bosonic components of (4.5)-(4.7) correspond to the equations of motion for A_1, B_2 and ϕ_0 respectively. This is clearly no surprise since we know that the B.I.'s (4.2)-(4.4), under a suitable choice of constraints, set the theory on-shell. Therefore, our procedure for dualizing the abelian connections can also be regarded as an alternative way for deriving their equations of motion.

4.1 The dual superconnections

The identities (4.5)-(4.7) allow now the introduction of dual potentials $\tilde{B}_2$, $\tilde{A}_3$, $\tilde{\phi}_4$ according to

$$\tilde{H} = d\tilde{B}, \tag{4.18}$$
$$\tilde{F} = d\tilde{A} + \tilde{B}F, \tag{4.19}$$
$$\tilde{V} = d\tilde{\phi} + H\tilde{B} + c_1 F\tilde{A}. \tag{4.20}$$

The possibility of describing the three-form H in terms of a B_2 potential or a dual $\tilde{B}_2$ potential in compatibility with supersymmetry is of course known from a long time[17].

The existence of dual $\tilde{A}_3$ potentials for Maxwell fields has been conjectured by Schwarz and Sen [24] in ten dimensions (in which case they become 7-superforms) in relation with the existence of a manifestly $SL(2,\mathbb{R})_S$ invariant form for the heterotic string effective action compactified down to four dimensions. These seven-superform gauge fields have actually been constructed in ten-dimensional superspace [10] so that the existence of their

six dimensional counterpart (three-superforms) in (4.19) constitutes actually no surprise: they could also have been obtained by compactifying $N = 1$, $D = 10$ SUGRA-MAXWELL toroidally down to six dimensions and then truncating the resulting $N = 2$, $D = 6$ theory to an $N = 1$ supersymmetric SUGRA-MAXWELL-TENSOR theory. The construction we gave here, however, is direct and independent of the details of any compactification scheme.

The significance of the supersymmetric dualization of the dilaton is, on the other hand, less clear. To our present knowledge the dual potential $\tilde{\phi}_4$ appearing in (4.20) did not have any direct application but it may be, for instance, that in some supersymmetric three-brane σ-model it can be coupled "naturally" to the brane through its pull-back on the four-dimensional brane world-volume.

There are, however, important differences between the features of $\tilde{B}_2$ on one hand, and the features of $\tilde{A}_3$ and $\tilde{\phi}_5$ on the other. If one uses $\tilde{B}_2$ to describe the graviphoton and A_1 and ϕ_0 to describe respectively the Maxwell fields and the dilaton (the standard dual $N = 1$, $D = 6$ SUGRA [17]) then in all the equations of motion the potentials appear only through their curvatures $\tilde{H}_3$, F_2 and V_1, and the theory can be described in terms of an action involving $\tilde{B}_2$, A_1 and ϕ_0.

If, on the other hand, one tries to describe the theory in terms of $\tilde{B}_2$, $\tilde{A}_3$ and ϕ_0, then one is faced with the problem of eliminating from all the equations of motion A_1 in favour of $\tilde{A}_3$; in particular in (4.19) the curvature F, which has now to be viewed as the dual of $\tilde{F}$, appears on the r.h.s. so that this equation determines $\tilde{F}$ in terms of $\tilde{A}_3$ only implicitly and it is not possible to write an action, at least in closed form, in which A_1 has been replaced by $\tilde{A}_3$. The situation is even worse when one tries to use $\tilde{\phi}_4$ instead of ϕ_0 using (4.20): in this case the dilaton ϕ_0 appears at the r.h.s. of (4.20) explicitly, i.e. not through its curvature $V = d\phi_0$, and it is not possible, not even implicitly, to eliminate ϕ_0 completely from the game in favour of $\tilde{\phi}_4$. Nevertheless, at the level of equations of motion, (4.18)-(4.20) are perfectly consistent with (4.5)-(4.7). For example, gauge invariance for $\tilde{B}$ in (4.19), $\tilde{B} \to \tilde{B} + dC$, is saved, upon using $dF = 0$, by imposing $\tilde{A} \to \tilde{A} - CF$.

5 Introducing the Hypermultiplets

Supersymmetry in six-dimensions allows the existence of matter fields, i.e. of hypermultiplets. In this section we generalize our results concerning duality to the presence of these fields for the "ungauged" case, i.e. when they are charge-less with respect to the abelian Maxwell fields.

As shown in [15, 25] and [26] self-interacting hypermultiplets in a $N = 1, D = 6$ theory live on a quaternionic Kähler manifold. For simplicity we will use the particular coset manifold $USp(n_H, 1)/USp(n_H) \otimes USp(1)$ where n_H is the number of hypermultiplets [15, 17]. In what follows the indices $I, J = 1, \ldots, 4n_H$ are used for the hyper-

scalars which parametrize the quaternionic manifold, $i,j=1,2$ are $USp(1)$ indices and $X,Y,Z = 1,\ldots,2n_H$ label the fundamental representation of $USp(n_H)$. $g_{IJ}(\phi)$ indicates the quaternionic manifold metric tensor, $\omega_{Ii}{}^j(\varphi)$ and $\omega_{IX}{}^Y(\varphi)$ are the $USp(1)$ and $USp(n_H)$ connections and e^I_{iZ} are the coset vielbeins. For completeness we remember also the relations obeyed by the vielbeins:

$$
\begin{aligned}
g_{IJ}\, e^I_{iX}\, e^J_{jZ} &= \epsilon_{ij}\,\epsilon_{XZ}, && (5.1)\\
e^I_{iZ}\, e^{JjZ} + e^J_{iZ}\, e^{IjZ} &= g^{IJ}\,\delta_i{}^j, && (5.2)\\
e^I_{iY}\, e^{JiZ} + e^J_{iY}\, e^{IiZ} &= \frac{1}{n_H}\, g^{IJ}\delta_Y{}^Z, && (5.3)
\end{aligned}
$$

where ϵ_{ij} and ϵ_{XY} are the invariant tensors of $USp(1)$ and $USp(n_H)$ respectively.

As we saw in section three the only freedom left in the solution of the torsion B.I.'s were the three spinors $\omega_{\alpha i,jk}$ in $T_{\alpha i\beta j}{}^{\gamma k}$. A natural choice for the introduction of hypermultiplets [17] amounts to the identification of $\omega_{\alpha i,jk}$ with the $USp(1)$ connection. This connection, which realizes local $USp(1)$ covariance, is a function of the hypermultiplet superscalars $\varphi^I(Z)$

$$\omega_{ij}(\varphi) = d\varphi^I \omega_{I\,ij}(\varphi), \tag{5.4}$$

its pull-back on the cotangent bundle basis of superspace being given by

$$\omega_{ij} = E^A D_A\varphi^I\omega_{I\,ij} \equiv E^{\alpha k}\omega_{\alpha k,ij} + E^a\omega_{a\,ij}. \tag{5.5}$$

The new torsion constraint would now read

$$T_{\beta j\gamma k}{}^{\alpha i} = \delta^\alpha_\beta\omega_{\gamma k,j}{}^i + (\beta j \leftrightarrow \gamma k) \tag{5.6}$$

and one should solve the torsion B.I.'s with this new dynamical constraint [17].

It is, however, more convenient to proceed in a slightly different, but equivalent, way. Instead of imposing the new constraint (5.6) on the torsion, which would then turn out to transform under $USp(1)$ as a connection and not as vector, we define a new torsion which is $USp(1)$ covariant and satisfies, moreover, the old constraint, $T_{\alpha i\beta j}{}^{\gamma k} = 0$:

$$
\begin{aligned}
T^a &\equiv dE^a + E^b\Omega_b{}^a, && (5.7)\\
T^{\alpha i} &\equiv \mathcal{D}E^{\alpha i} = dE^{\alpha i} + E^{\beta i}\Omega_\beta{}^\alpha + E^{\alpha j}\omega_j{}^i. && (5.8)
\end{aligned}
$$

It is also convenient to define a $USp(1)$ and $USp(n_H)$ covariant derivative $\mathcal{D}_A$ through

$$\mathcal{D}_A \;=\; D_A + D_A\varphi^I\,\omega_{I\#}{}^{\#} \tag{5.9}$$

where ω_I equals $\omega_{Ii}{}^j$ or $\omega_{IX}{}^Y$ or both of them according to the tensor on which it acts.

With respect to the standard procedure our procedure has the advantage of being manifestly $USp(1)$ and $USp(n_H)$ covariant in that the connections appear in the B.I.'s and their solutions only through the covariant derivatives.

Closure of the SUSY algebra on the hypermultiplets entails

$$D_{\alpha i}\varphi^I = e^I_{iZ}\psi^Z_\alpha, \tag{5.10}$$

$$\mathcal{D}_{\alpha i}\psi^Z_\beta = 2\, e^Z_{iI}\,\Gamma^a_{\alpha\beta}\, D_a\varphi^I, \tag{5.11}$$

where the hyperfermions ψ^Z_α are the supersymmetric partners of φ^I. This implies in particular that $\omega_{\alpha i,j}{}^k = D_{\alpha i}\varphi^I\omega_{Ij}{}^k = e^I_{iZ}\psi^Z_\alpha\omega_{Ij}{}^k$.

Once all derivatives have been covariantized, the introduction of the hypermultiplets in our formalism leads essentially only to a change of the torsion B.I.'s themselves. The definitions (5.7), (5.8) give, in fact, rise to the new B.I.'s

$$DT^a = E^b R_b{}^a, \tag{5.12}$$

$$\mathcal{D}T^{\alpha i} = E^{\beta i}R_\beta{}^\alpha + E^{\alpha j}\mathcal{R}_j{}^i, \tag{5.13}$$

where $\mathcal{R}_j{}^i = d\omega_j{}^i + \omega_j{}^k\omega_k{}^i = 1/2\, d\varphi^I\, d\varphi^J\, \mathcal{R}_{JI\,j}{}^i$ is the $USp(1)$ curvature. As is well known SUSY requires the quaternionic manifold to be maximally symmetric, that is, in our conventions

$$\mathcal{R}_{IJij} = 2(e_{IiZ}e_{Jj}{}^Z - e_{JiZ}e_{Ij}{}^Z). \tag{5.14}$$

The new torsion B.I.'s can now again be consistently solved with the basic constraints

$$T^a_{\alpha i\beta j} = -2\epsilon_{ij}\Gamma^a_{\alpha\beta}, \tag{5.15}$$

$$T_{\alpha i\beta j}{}^{\gamma k} = 0 = T_{a\alpha i}{}^b. \tag{5.16}$$

The differences with respect to the case without hypermultiplets arise now primarily from the new term on the r.h.s. of (5.13), i.e. $E^{\alpha j}\mathcal{R}_j{}^i$; all new terms are however *manifestly* covariant.

Taking (5.11) and (5.14) into account one finds for the torsion and $SO(1,5)$–curvature:

$$T_{abc} = T^-_{abc} - \frac{1}{4}\psi^+_{abc}, \tag{5.17}$$

$$T_{a\alpha i}{}^\beta_j = 3\delta_\alpha{}^\beta V_{aij} + \Gamma_{ac\,\alpha}{}^\beta V^c_{ij} - \frac{1}{4}\epsilon_{ij}\Gamma^{bc}{}_\alpha{}^\beta\left(T^-_{abc} - \frac{1}{4}\psi^+_{abc}\right), \tag{5.18}$$

$$R_{\alpha i\beta jab} = -4\Gamma_{abc\,\alpha\beta}V^c_{ij}. \tag{5.19}$$

where the self-dual tensor ψ^+_{abc} is defined by

$$\psi^+_{abc} \equiv \psi_Y\Gamma_{abc}\psi^Y. \tag{5.20}$$

For the YM– and tensor–multiplets the relevant new SUSY transformations are now

$$\mathcal{D}_{\alpha i}\chi^\beta_j = \delta_\alpha{}^\beta A_{ij} - \epsilon_{ij}\frac{1}{4}\Gamma^{ab}{}_\alpha{}^\beta F_{ab}, \tag{5.21}$$

$$\mathcal{D}_{\alpha i}F_{ab} = 4\Gamma_{[b\ \alpha\rho}\mathcal{D}_{a]}\chi^\rho_i + 4(\Gamma_{ba}\Gamma_c)_{\alpha\rho}V^c_{ij}\chi^{\rho j} + \frac{1}{4}\Gamma^{cd}{}_{[b\,\alpha\rho}\psi^+_{a]cd}\chi^\rho_i, \tag{5.22}$$

$$\mathcal{D}_{\alpha i}\lambda_{\beta j} = 4\phi\Gamma_{c\alpha\beta}V^c_{ij} + c_1\Gamma_{c\alpha\beta}\chi_i\Gamma^c\chi_j + \epsilon_{ij}\Gamma^a_{\alpha\beta}D_a\phi - \frac{1}{12}\epsilon_{ij}\Gamma^{abc}_{\alpha\beta}\left(H^+_{abc} + \frac{\phi}{4}\psi^+_{abc}\right) \tag{5.23}$$

while in particular the relation between H^-_{abc} and T^-_{abc} is unchanged. The consistency conditions remain formally the same

$$\mathcal{D}_{\alpha(i}V^a_{jk)} = 0, \tag{5.24}$$

$$\mathcal{D}_{\alpha(i}A_{jk)} = -4\Gamma^a_{\alpha\beta}\chi^\beta_{(i}V_{a\ jk)}, \tag{5.25}$$

with the simple change $D_{\alpha i} \to \mathcal{D}_{\alpha i}$ with respect to equations (3.32) and (3.33), and they are again solved by equations (3.41).

Now it is easy to solve the B.I.'s for the dual superforms $\tilde{H}$, $\tilde{F}$ and $\tilde{V}$. Indeed, these dual supercurvatures are again given by (4.8)-(4.17) with the *unique* difference that now

$$\tilde{H}_{abc} = \frac{1}{\phi^2}\frac{1}{3!}\epsilon_{abcdef}\left(H^{def} - \frac{1}{4\phi}\lambda_i\Gamma^{def}\lambda^i - \frac{\phi}{2}\psi^{+def} + c_1 tr(\chi^i\Gamma^{def}\chi_i)\right). \tag{5.26}$$

6 Non minimal models: the Lorentz Chern-Simons form

The dimensions $D = 2, 6, 10$ are special in many respects. They allow for example, in a space-time with Minkowskian signature, the existence of chiral bosons with self-dual curvatures, respectively one-, three- and five-forms.

They are also the unique dimensions below eleven which are potentially plagued by Lorentz anomalies. In all physically significant theories, however, the total anomaly polynomial factorizes and the anomaly can be cancelled via the Green-Schwarz mechanism [27]. The essential ingredients are in each case a modified B.I. for a generalized three-form curvature, in ordinary bosonic space,

$$dH = c_1 trF^2 + c_2 trR^2, \tag{6.1}$$

(a classical effect) and the subtraction from the effective action of a local counterterm, proportional to B_2 (a quantum effect). Both ingredients, however, do not directly respect supersymmetry. Since (6.1) is a local and classical relation its supersymmetrization can be achieved by solving the corresponding B.I. in *superspace*. The analogous problem in $N = 1, D = 10$ supergravity has been solved in [12, 13] with the aid of the Bonora-Pasti-Tonin [BPT] theorem. The aim of the present section is to "extend" this theorem to six-dimensional supergravity and to solve (6.1) for $c_2 \neq 0$ in superspace. At the level of the action the Lorentz–Chern–Simons form gives rise to non-minimal couplings of the form $(R_{abcd})^2$, together with their supersymmetric completion.

Having solved (6.1), in the next section we will discuss the existence of the dual superpotentials $\tilde{B}$, $\tilde{A}$ and $\tilde{\phi}$ in the resulting non-minimal $N = 1$, $D = 6$ supergravity.

For the sake of simplicity we turn back to the hypermultiplet free model considered in section three. For convenience we repeat here the general parametrizations of the torsion

and curvatures we obtained:

$$
\begin{aligned}
T_{\alpha i\beta j}{}^{a} &= -2\epsilon_{ij}\Gamma^{a}_{\alpha\beta}, && (6.2)\\
T_{\alpha ia}{}^{b} &= 0 = T_{\alpha i\beta j}{}^{\gamma k}, && (6.3)\\
T_{abc} &= T^{-}_{abc}, && (6.4)\\
T_{a\alpha i}{}^{\beta}_{j} &= 3\delta_{\alpha}{}^{\beta}V_{aij} + \Gamma_{ac\alpha}{}^{\beta}V^{c}_{ij} - \epsilon_{ij}\frac{1}{4}\Gamma^{bc}{}_{\alpha}{}^{\beta}T^{-}_{abc}, && (6.5)\\
R_{\alpha i\beta jab} &= -4\Gamma_{abc\,\alpha\beta}V^{c}_{ij}, && (6.6)\\
R_{a\alpha ibc} &= -2\Gamma_{a\,\alpha\beta}T^{\beta}_{bci} + \Gamma_{[ab\,\alpha}{}^{\beta}\left(\frac{2}{3}\epsilon^{hk}\Gamma_{c]s\beta}{}^{\rho}D_{\rho h}V^{s}_{ki} - \frac{4}{3}\epsilon^{hk}D_{\beta h}V_{c]ki}\right), && (6.7)\\
D_{\alpha i}T^{-}_{abc} &= 6(\Gamma_{[a})_{\alpha\beta}T^{\beta}_{bc]i} - 2\Gamma_{[ab\alpha}{}^{\beta}\left(\frac{2}{3}\epsilon^{hk}\Gamma_{c]s\beta}{}^{\rho}D_{\rho h}V^{s}_{ki} - \frac{4}{3}\epsilon^{hk}D_{\beta h}V_{c]ki}\right), && (6.8)\\
D_{\alpha i}V^{a}_{jk} &= \epsilon_{i(j}X_{k)\alpha}. && (6.9)
\end{aligned}
$$

We included here in the last equation also the consistency condition for the unknown auxiliary field V^{a}_{ij}; at the end we will obtain an (implicit) expression for this field in terms of the physical fields, see (6.63) below, and one has to check if it satisfies (6.9).

BPT Theorem. The parametrizations (6.2)-(6.9) allow for the decomposition

$$
trR^{2} = dX + K, \qquad (6.10)
$$

where X and K are gauge invariant three and four-superforms respectively and

$$
K_{(0,4)} = 0 = K_{(1,3)}. \qquad (6.11)
$$

A simple consequence of $dtrR^{2} = 0$ is that $dK = 0$. Before proving the theorem we remark that if we define

$$
\hat{H} = H - c_{2}X \qquad (6.12)
$$

the H-B.I. becomes

$$
d\hat{H} = c_{1}trF^{2} + c_{2}K. \qquad (6.13)
$$

This identity can now be solved imposing on $\hat{H}$ the "old" constraints

$$
\begin{aligned}
\hat{H}_{\alpha i\beta j\gamma k} &= 0, && (6.14)\\
\hat{H}_{a\alpha i\beta j} &= -2\phi\epsilon_{ij}(\Gamma_{a})_{\alpha\beta}, && (6.15)\\
\hat{H}_{\alpha iab} &= -\Gamma_{ab\,\alpha}{}^{\beta}\lambda_{\beta i} && (6.16)
\end{aligned}
$$

since the four-form K has the same structure as trF^{2}: $d\ trF^{2} = 0$, $(trF^{2})_{(0,4)} = (trF^{2})_{(1,3)} = 0$.

For the proof of the theorem we adopt the techniques used in [12]. We write the superspace differential d (when it acts on Lorentz *invariant* forms) as a sum of operators

$$
d = \overline{d} + \overline{D} + T + \tau \qquad (6.17)
$$

each of which sends a (p,q)-superform to a (p',q')-superform. Defining their degree as the difference $(p'-p, q'-q)$, the operators $\overline{d}$, $\overline{D}$, T, τ have respectively degree $(1,0)$, $(0,1)$, $(-1,2)$ and $(2,1)$.

In more detail: $\overline{d} = E^a D_a + T_{(1,0)}$, $\overline{D} = E^{\alpha i} D_{\alpha i} + T_{(0,1)}$, $T = T_{(-1,2)}$, $\tau = T_{(2,-1)}$; where D_a and $D_{\alpha i}$ are the ordinary covariant derivatives acting only on the components, while $T_{(r,s)}$ acts only on the vielbeins as follows:

$$T_{(-1,2)} E^a = \epsilon_{ij} \Gamma^a_{\alpha\beta} E^{\alpha i} E^{\beta j}, \qquad T_{(-1,2)} E^{\alpha i} = 0, \tag{6.18}$$

$$T_{(2,-1)} E^a = 0, \qquad T_{(2,-1)} E^{\alpha i} = \frac{1}{2} E^b E^c T_{cb}{}^{\alpha i}, \tag{6.19}$$

$$T_{(1,0)} E^a = \frac{1}{2} E^b E^c T_{cb}{}^a, \qquad T_{(1,0)} E^{\alpha k} = E^{\beta i} E^b T_{b\beta i}{}^{\alpha k}, \tag{6.20}$$

$$T_{(0,1)} E^a = 0, \qquad T_{(0,1)} E^{\alpha i} = 0. \tag{6.21}$$

Due to $d^2 = 0$ these operators satisfy the following anticommutation rules:

$$\begin{aligned} T^2 &= 0 \\ \overline{D}T + T\overline{D} &= 0 \\ \overline{D}^2 + \overline{d}T + T\overline{d} &= 0 \\ \overline{dD} + \overline{Dd} + T\tau + \tau T &= 0 \\ \overline{d}^2 + \overline{D}\tau + \tau\overline{D} &= 0 \\ \overline{d}\tau + \tau\overline{d} &= 0 \\ \tau^2 &= 0. \end{aligned} \tag{6.22–6.23}$$

A crucial ingredient of the proof is in particular the operator T which, as direct consequence of the cyclic identity, is a coboundary operator

$$T^2 = 0. \tag{6.24}$$

If we call $S_{(p,q)}$ the space of (p,q)-superforms, then T maps $S_{(p,q)}$ to $S_{(p-1,q+2)}$.

Setting $Q \equiv trR^2$, our starting point is the identity

$$dQ = 0. \tag{6.25}$$

Projecting it on the sectors $(p, 5-p)$, we obtain

$$TQ_{(0,4)} = 0, \tag{6.26}$$

$$TQ_{(1,3)} + \overline{D}Q_{(0,4)} = 0, \tag{6.27}$$

$$TQ_{(2,2)} + \overline{D}Q_{(1,3)} + \overline{d}Q_{(0,4)} = 0. \tag{6.28}$$

The identity (6.26) states that $Q_{(0,4)}$ is a T-cocycle; the first step is to verify that it is a trivial one. This means that there exists a $X_{(1,2)} \in S_{(1,2)}$ such that

$$Q_{(0,4)} = TX_{(1,2)}. \tag{6.29}$$

The explicit expression for $Q_{(0,4)}$ can be obtained from the expression of $R_{\alpha i\beta ja}{}^{b}$ (6.6)

$$Q_{(0,4)} = 4E^{\delta l}E^{\gamma k}E^{\beta j}E^{\alpha i}((\Gamma_{abc})_{\alpha\beta}(\Gamma^{bas})_{\gamma\delta}V^{c}_{ij}V_{s\ kl}) \tag{6.30}$$

and one easily finds that indeed

$$Q_{(0,4)} = T\overline{X}_{(1,2)} \tag{6.31}$$

with

$$\overline{X}_{(1,2)} = \frac{3}{2}E^{a}E^{\alpha k}\Gamma^{b}_{\alpha\beta}E^{\beta}_{k}V_{aij}V^{ij}_{b}. \tag{6.32}$$

Clearly $X_{(1,2)}$ is defined only modulo a non-trivial T-cocycle $Y_{(1,2)} \in S_{(1,2)}$ so that we can set more in general

$$X_{(1,2)} = \overline{X}_{(1,2)} + Y_{(1,2)}, \qquad TY_{(1,2)} = 0. \tag{6.33}$$

We neglect here trivial T-cocycles $\in S_{(1,2)}$ since they express simply the fact that X is defined modulo trivial cocycles of d itself which amount to a redefinition of B. The general non-trivial cocycles of T in $S_{(1,2)}$ are of the form

$$Y_{a\alpha i\beta j} = (\Gamma_{abc})_{\alpha\beta}Y^{bc}_{ij}, \qquad Y^{bc}_{ij} = Y^{[bc]}_{(ij)}, \tag{6.34}$$

if we disregard terms of the form

$$\epsilon_{ij}\Gamma_{a\alpha\beta}\hat{\phi} \tag{6.35}$$

which amount to a redefinition of ϕ. On dimensional grounds there are only three terms which can contribute to Y^{bc}_{ij}:

$$Y^{(1)}_{bc\,ij} = D_{[b}V_{c]ij} \tag{6.36}$$

$$Y^{(2)}_{bc\,ij} = V_{[b\,i}{}^{k}V_{c]\,jk} \tag{6.37}$$

$$Y^{(3)}_{bc\,ij} = T^{-}_{abc}V^{a}_{ij}. \tag{6.38}$$

Inserting now (6.29) in (6.27), we obtain

$$T(Q_{(1,3)} - \overline{D}X_{(1,2)}) = 0, \tag{6.39}$$

which means that $(Q_{(1,3)} - \overline{D}X_{(1,2)}) \in S_{(1,3)}$ is a T-cocycle. It can be seen that for a generic $Y_{(1,2)}$ it is a non–trivial one. An explicit computation reveals, in fact, that it becomes a trivial cocycle if and only if one chooses

$$Y^{bc}_{ij} = Y^{(3)}{}^{bc}_{ij} + \alpha\frac{1}{3}Y^{(1)}{}^{bc}_{ij} + (1-\alpha)Y^{(2)}{}^{bc}_{ij}, \tag{6.40}$$

where α is an arbitrary constant.

We will comment on the correct choice for the parameter α below. Here we would only like to remark the following. If $\alpha \neq 0$ then $Y_{(1,2)}$ would contain a term linear in the bosonic derivative of the still undetermined auxiliary field V^{a}_{ij}. The H-B.I. would then give rise to an equation for this field, see (6.63) below, of the kind

$$V^{a}_{ij} = c\,\Box V^{a}_{ij} + \dots, \tag{6.41}$$

where c is a constant, proportional to α, and the remaining terms are quadratic and of higher order in V^a_{ij}. This equation, which is the generalization of (3.41) to $c_2 \neq 0$, would now propagate unphysical spurious degrees of freedom due to the appearance of the D'Alambertian. It can be seen that these degrees of freedom are actually unitary-violating poltergeists. The presence of such poltergeists in supergravity theories with Lorentz-Chern-Simons terms is indeed known from a long time. Their supersymmetric structure, their meaning and their possible elimination, has been discussed in detail in [12], to which we refer the reader for more details.

With the choice (6.40), $Q_{(1,3)} - \overline{D}X_{(1,2)}$ becomes a trivial T-cocycle and the equation

$$Q_{(1,3)} = TX_{(2,1)} + \overline{D}X_{(1,2)} \tag{6.42}$$

determines the three-form $X_{(2,1)}$ uniquely since in the sector $(2,1)$ there are no T- cocycles at all:

$$TX_{(2,1)} = 0 \Leftrightarrow X_{(2,1)} = 0. \tag{6.43}$$

The explicit expression for $X_{(2,1)}$ will not be needed in what follows however.

Inserting now (6.29) and (6.42) in (6.28) and using the anticommutation properties of $T, \overline{D}, \overline{d}$ (6.22) one gets

$$T(Q_{(2,2)} - \overline{D}X_{(2,1)} - \overline{d}X_{(1,2)}) \equiv TW_{(2,2)} = 0. \tag{6.44}$$

The non-trivial T-cocycles in the $(2,2)$ sector have the general structure

$$K_{(2,2)} = E^a E^b E^{\alpha i} E^{\beta j}(-(\Gamma_a)_{\alpha\gamma}(\Gamma_b)_{\beta\delta} L^{\gamma\delta}_{ij}), \tag{6.45}$$

where $L^{\gamma\delta}_{ij} = L^{[\gamma\delta]}_{(ij)}$, and therefore

$$W_{(2,2)} = TX_{(3,0)} + K_{(2,2)}. \tag{6.46}$$

Again, the forms $X_{(3,0)}$ and $K_{(2,2)}$, i.e. $L^{\gamma\delta}_{ij}$, are uniquely determined by the above formulae, but their explicit expressions would turn out to be rather complicated and we do not need them here. In conclusion, we got the last decomposition needed to prove the six-dimensional version of the BPT theorem:

$$Q_{(2,2)} = TX_{(3,0)} + \overline{D}X_{(2,1)} + \overline{d}X_{(1,2)} + K_{(2,2)}. \tag{6.47}$$

Indeed, formulae (6.29),(6.42) and (6.47) combine now exactly to give (6.10) if one defines

$$X = X_{(1,2)} + X_{(2,1)} + X_{(3,0)}. \tag{6.48}$$

¿From these formulae one sees also that $K_{(0,4)} = K_{(1,3)} = 0$ and that $K_{(2,2)}$ is given in (6.45). Notice that at this point the knowledge of $K_{(2,2)}$ and the identity $dK = 0$ determine the four-superform K uniquely (use again the lemma). Notice also that $K_{(2,2)}$, given in (6.45), has indeed the same structure as $(trF^2)_{(2,2)}$.

Q.E.D.

6.1 Consistency conditions

We did not really end our proof of the compatibility between the Lorentz–Chern–Simons form and supersymmetry. We still have to show that the H-B.I. can be solved consistently and that the consistency conditions (3.32)-(3.33) are satisfied. To this end, as anticipated, we define $\hat{H} = H - c_2 X$, whose B.I. reads $d\hat{H} = c_1 trF^2 + c_2 K$, i.e.

$$
\begin{aligned}
(d\hat{H})_{(0,4)} &= 0 && (6.49)\\
(d\hat{H})_{(1,3)} &= 0 && (6.50)\\
(d\hat{H})_{(2,2)} &= c_1(trF^2)_{(2,2)} + c_2 K_{(2,2)} && (6.51)\\
(d\hat{H})_{(3,1)} &= c_1(trF^2)_{(3,1)} + c_2 K_{(3,1)} && (6.52)\\
(d\hat{H})_{(4,0)} &= c_1(trF^2)_{(4,0)} + c_2 K_{(4,0)} && (6.53)
\end{aligned}
$$

and, due to the lemma, it is sufficient to solve (6.49)-(6.51). On $\hat{H}$ we impose the constraints

$$
\begin{aligned}
\hat{H}_{\alpha i\beta j\gamma k} &= 0, && (6.54)\\
\hat{H}_{a\alpha i\beta j} &= -2\phi\epsilon_{ij}(\Gamma_a)_{\alpha\beta}, && (6.55)\\
\hat{H}_{ab\alpha i} &= -(\Gamma_{ab})_\alpha{}^\beta\lambda_{\beta i}, && (6.56)
\end{aligned}
$$

which solve already (6.49)-(6.50). Equation (6.51) implies (and is solved) by the relations:

$$
\begin{aligned}
D_{\alpha i}\lambda_{\beta j} &= 4\phi\Gamma_{c\alpha\beta}V^c_{ij} - 2(c_1\chi_{\alpha\beta\, ij} + c_2 L_{\alpha\beta\, ij}) + \epsilon_{ij}(\Gamma^a_{\alpha\beta}D_a\phi - \frac{1}{12}\hat{H}^+_{\alpha\beta}). && (6.57)\\
T^-_{abc} &= \frac{1}{\phi}\hat{H}^-_{abc} + \frac{c_1}{\phi}tr(\chi_i\Gamma_{abc}\chi^i), && (6.58)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_{\alpha\beta\, ij} &= -\Gamma^a_{\alpha\beta}tr(\chi_i\Gamma_a\chi_j), && (6.59)\\
L_{\alpha\beta\, ij} &= -\Gamma_{a\,\alpha\beta}(\Gamma^a_{\gamma\delta}L^{\gamma\delta}_{ij}). && (6.60)
\end{aligned}
$$

We remain finally only with the consistency conditions:

$$
\begin{aligned}
D_{\alpha(i}A_{jk)} &= 4\Gamma^a_{\alpha\beta}\chi^\beta_{(i}V_{a\, jk)}, && (6.61)\\
D_{\alpha(i}V^a_{jk)} &= 0. && (6.62)
\end{aligned}
$$

Maintaining for A_{ij} the definition in (3.41), the first condition tells us that V^a_{ij} has to satisfy the equation

$$
V^a_{ij} = -\frac{1}{16\phi^2}\lambda_i\Gamma^a\lambda_j - \frac{1}{16\phi}\Gamma^{a\,\alpha\beta}(c_1\chi_{\alpha\beta ij} + c_2 L_{\alpha\beta ij}). \qquad (6.63)
$$

Let us stress that even if the super YM-fields are absent ($F = 0$), in which case the consistency condition (6.61) becomes trivial, this equation is implied anyway by the closure of the SUSY algebra on $\lambda_{\alpha i}$.

This equation determines V_{ij}^a only implicitly in that $L_{ij}^{\gamma\delta}$ is a complicated expression which involves V_{ij}^a itself in a non polynomial way (a part from the linear term discussed in (6.41)).

The second condition instead implies a new constraint on the L supersymmetry transformation [2]:

$$D_\alpha^{(i}L_{\beta\gamma}^{jk)} = \frac{3}{\phi}\lambda_{[\alpha}^{(i}L_{\beta\gamma]}^{jk)}. \tag{6.64}$$

The identity $dK = 0$ ensures that $D_\alpha^{(i}L_{\beta\gamma}^{jk)}$ is antisymmetric in $\alpha\beta\gamma$

$$D_\alpha^{(i}L_{\beta\gamma}^{jk)} = D_{[\alpha}^{(i}L_{\beta\gamma]}^{jk)} \tag{6.65}$$

which is the right structure required by (6.64), but we did not perform the long calculations needed to prove that $D_{[\alpha}^{(i}L_{\beta\gamma]}^{jk)}$ equals exactly $\frac{3}{\phi}\lambda_{[\alpha}^{(i}L_{\beta\gamma]}^{jk)}$.

These calculations would first of all require the explicit expression for $X_{(2,1)}$ which can be obtained from (6.42); this could then be inserted in (6.47) to compute the explicit formula for $L_{ij}^{\alpha\beta}$. The algebraic manipulation involved are straightforward, from a technical point of view, but very lengthy. So this is the missing point in our proof of the compatibility of the Lorentz-Chern-Simons form with SUSY.

Let us notice however that to satisfy (6.65) we have still the freedom to choose the parameter α in (6.40) arbitrarily. Indeed, since the poltergeists are present also in $N = 1$, $D = 10$ SUGRA with a Lorentz-Chern-Simons term, and the present theory could be obtained from the ten-dimensional one upon compactification and truncation from $N = 2$ to $N = 1$ SUSY, we expect a non-vanishing α.

Even if (6.64) can not be satisfied for any α, our procedure leaves another possibility open. In fact, as has been noted for the ten-dimensional case [12], the decomposition $trR^2 = dX + K$ is not unique. Suppose, in fact, that there exist non-exact three-superforms Z satisfying

$$(dZ)_{(0,4)} = 0 = (dZ)_{(1,3)}. \tag{6.66}$$

Then one can write

$$trR^2 = d\hat{X} + \hat{K}, \tag{6.67}$$

where now $\hat{X} = X + Z$ and $\hat{K} = K - dZ$, and $\hat{K}$ shares the same good properties with K, i.e. satisfies

$$\hat{K}_{(0,4)} = 0 = \hat{K}_{(1,3)} \tag{6.68}$$

$$d\hat{K} = 0. \tag{6.69}$$

It could then be that only for a suitable choice of Z one may satisfy (6.64). The problematic feature of such a solution lies in the fact that the three superform Z has to be

[2]This constraint is identically satisfied by $\chi_{\alpha\beta ij}$

constructed using explicitly the fields of the tensor multiplet, whose SUSY transformations are not known from the beginning. In fact, the combinations which involve only the fields appearing in the parametrizations (6.2)-(6.9) have – for dimensional reasons – already been exhausted by our "minimal" decomposition of trR^2. Therefore, if a solution can be achieved only for a non vanishing Z it can presumably not be obtained in a closed form.

6.2 Duality in the presence of a Lorentz-Chern-Simons form

We suppose in this section that (6.64) holds indeed and analyze again the possibility of dualizing the connections ϕ, A and B in the non-minimal model constructed in the preceding section.

The answer is very simple and positive for what concerns $\tilde{H}$ and the (abelian) $\tilde{F}$. The B.I.'s (4.5) and (4.6) are again satisfied if one defines $\tilde{F}$ exactly as in (4.12)-(4.14) while for $\tilde{H}$ one can again use (4.8)-(4.11) with the only difference that now

$$\tilde{H}_{abc} = \frac{1}{\phi^2}\frac{1}{3!}\epsilon_{abcdef}\left(\hat{H}^{def} - \frac{1}{4\phi}\lambda_i\Gamma^{def}\lambda^i + c_1 tr(\chi^i\Gamma^{def}\chi_i)\right). \tag{6.70}$$

On the other hand the dilaton has now to be described as a zero-form, i.e. as a scalar, since the B.I. for $\tilde{V}$ can be no longer (4.7) in that the r.h.s. of (4.7) is not closed anymore. This is due to the modified B.I. for H, $dH = c_1 trF^2 + c_2 trR^2$, and to the fact that the Lorentz curvature $R_a{}^b$ cannot be dualized for its intrinsic non-abelian nature.

7 Some concluding remarks

The variety of couplings in supergravity theories becomes more and more restricted as the dimensionality increases. Eleven-dimensional supergravity allows only for the pure SUGRA multiplet and only the minimally coupled theory has been explicitly constructed [28]. In ten dimensions there is, in addition to the SUGRA multiplet, the YM multiplet (in the case of $N = 1$ SUSY) and in this case, in addition to the minimally coupled theory, there exists also an exact solution for anomaly free $N = 1$, $D = 10$ SUGRA-SYM, i.e. in the presence of a Lorentz-Chern-Simons form [12, 13].

For these dimensions the supersymmetric dualization of the abelian connections has been carried out in [8, 10, 11, 13]. The results of those papers indicated that the existence of the supersymmetric duals of the abelian connections constitutes actually a general feature of all supergravity theories, independently of their dimensionality, even if a general proof of this statement is still missing. The absence of such a general theorem is probably related to the fact that Hodge–duality can not be canonically lifted to superspace.

In the present paper we extended this analysis to the six-dimensional case. The essentially new features of six-dimensional supergravity theories, w.r.t. 10 and 11 dimensions, are constituted by the appearance of matter (hyper- and tensor)-multiplets which enlarge

significantly the possible couplings between pure supergravity and the other multiplets. Our general analysis revealed, however, that these couplings are restricted by certain consistency conditions, which are clearly satisfied for minimal couplings, and also in the anomaly free non-minimal theory modulo eq. (6.64) which needs still to be checked. For what concerns duality we were able to confirm the expectation which emerged from ten and eleven dimensions: the B_2 form and the abelian Maxwell fields can be dualized in both cases, while the dilaton can be dualized only in the minimally coupled theory. The possible relevance of the dual Maxwell fields has already been noticed [10, 24] while possible applications of the dualized dilaton are still outstanding.

Possible generalizations of these results could be gained in the following directions. Recently rather general couplings of an arbitrary number n_T of tensor-multiplets have been worked out in [4]. Since in this case one B_2 curvature (essentially the one belonging to the SUGRA-multiplet) is anti-selfdual, while the remaining n_T ones are self-dual, the B_2 curvatures can no longer be dualized while it should still be possible to dualize the 'dilatons' and the abelian Maxwell fields.

Another interesting point concerns the dualizability of the hyperscalars. In a globally supersymmetric (and free) theory they can actually be described as four-form potentials, as has been shown long time ago in [18], while it seems unlikely that they can be dualized in supergravity due to the quaternionic structure of the underlying manifold.

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Appendix: Notations and Conventions

We write our symplectic Majorana-Weyl spinors as $\psi_{\alpha i}$ (left-handed) and $\psi^{\alpha i}$ (right-handed) where $i = 1, 2$ is an $USp(1)$ index which can be raised and lowered with the invariant antisymmetric tensor ϵ_{ij}

$$\psi_i = \epsilon_{ij}\psi^j, \qquad \psi^i = \epsilon^{ji}\psi_j, \tag{A.1}$$

while α is a chiral $SO(1,5)$ spinor index which cannot be raised or lowered. The symplectic Majorana-Weyl condition reads

$$\epsilon^{ij}\psi^{\alpha j} = O^{\alpha\beta}\psi^{\star\beta i} \tag{A.2}$$

where the matrix O satisfies

$$O^T = -O, \quad O^\star = O, \quad O^2 = -\mathbb{I}. \tag{A.3}$$

The matrices $(\Gamma^a)_{\alpha\beta}$ and $(\Gamma^a)^{\alpha\beta}$ span a Weyl-algebra, $(\Gamma_{(a})_{\alpha\beta}(\Gamma_{b)})^{\beta\gamma} = \eta_{ab}\delta_\alpha^\gamma$, and satisfy the hermiticity condition

$$O\Gamma^{a\dagger}O = \Gamma^a. \tag{A.4}$$

Notice, however, that the relations (A.2)–(A.4) need never used explicitly.

The duality relations for the anti-symmetrized Γ-matrices is

$$(\Gamma_{a_1\ldots a_k})_{\alpha\beta} = -(-1)^{k(k+1)/2}\frac{1}{(6-k)!}\epsilon_{a_1\ldots a_6}\Gamma^{a_{k+1}\ldots a_6}_{\alpha\beta} \tag{A.5}$$

where no "γ_7" appears since our Γ-matrices are 4×4 Weyl matrices, and the cyclic identity reads

$$(\Gamma^a)_{\alpha(\beta}(\Gamma_a)_{\gamma)\delta} = 0. \tag{A.6}$$

Another fundamental identity is

$$(\Gamma_a)_{\alpha\beta}(\Gamma^a)^{\gamma\delta} = -4\delta^\gamma_{[\alpha}\delta^\delta_{\beta]}. \tag{A.7}$$

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